# Column Number Density Expressions Through $M$ = 0 and $M$ = 1 Point Source Plumes Along Any Straight Path

Michael Woronowicz
SGT, Inc.

CCMPP 17, NASA Goddard Space Flight Center
18-20 July 2017

# Outline

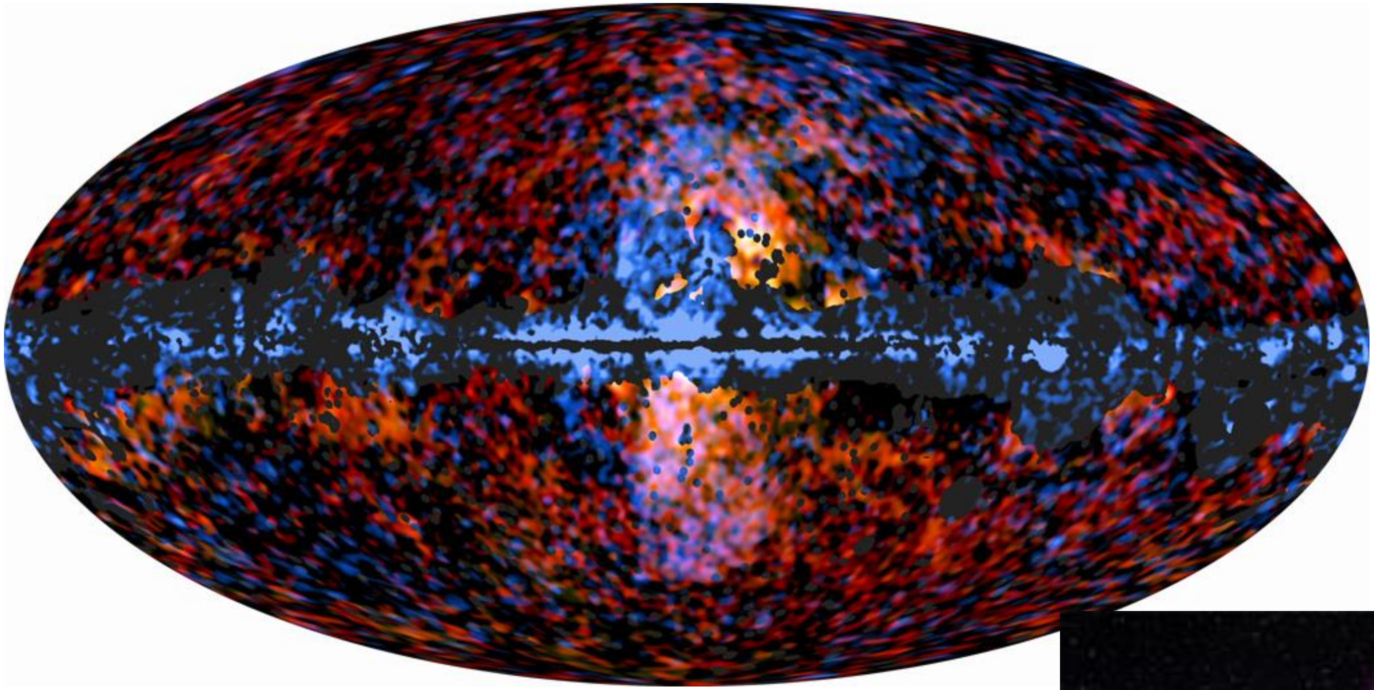
- Introduction
- Objective
- Venting Source Model
  - Results for $M = 0$ Cases
    - 1-D, 2-D, 3-D
  - Approximate $M = 1$ Angular Distribution
  - Results for $M = 1$ Cases
    - 1-D, 2-D, 3-D
  - $M > 1$ Observation
- Unconstrained Radial Source Model
  - Results for $M = 0$ Cases
- Concluding Remarks

# ATV Edoardo Amaldi Approaches ISS

# Introduction

- Providers of externally-mounted scientific payloads at the International Space Station (ISS) are required to evaluate column number density (CND, $\sigma$) associated with various gas releases and demonstrate that they fall below some maximum requirement
  - Must be considerate of other payloads
  - Since this includes unknown future additions, becomes a search for maximum CND along any path
- Occasionally astrophysicists are interested in estimating the amount of gas released by some event or process by evaluating light attenuation of a distant star having known properties due to this release
  - Milky Way center, black hole, “Fermi Bubbles”

# “Fermi Bubbles”



ESA/Planck Collaboration (microwave); NASA/DOE/Fermi LAT/Dobler et al./Su et al. (gamma rays)

# Objective

- Develop analytical CND expressions for general paths that intercept various common point sources under high vacuum conditions
  - Effusion/low rate evaporation/outgassing ($M = 0$)
  - Venting via sonic orifice ($M = 1$)
  - Spherically-symmetric, radial expansion ($M = 0$)

# Venting (Directed) Source Behavior

- External neutral gas phase sources on ISS result from a number of different physical mechanisms
  - Supersonic expansion through thruster nozzles
  - Pressure-driven acceleration to sonic conditions across an orifice
  - Surface evaporation, desorption (may or may not have bulk velocity)
  - Effusion--low-rate, high-$Kn$ venting ($M = 0$)
  - Diffusion-limited outgassing ($M = 0$)
- This study assumes that, for these applications, the point source may be described using free molecule flow model approximations
  - Density levels fall rapidly with distance from source location
  - Existence of self-scattering collisions may not substantially alter plume distribution from free molecule flow description

# Directed Source—Steady Density

- Can compute many different types of local quantities at receiver position ***x*** relative to source
  - Steady number density $n$ from a directed axisymmetric source given by

$$n(\mathbf{x}) = \frac{\beta \dot{N} \cos\theta}{A_1 \pi r^2} e^{w^2 - s^2} \left\{ w e^{-w^2} + \left( \frac{1}{2} + w^2 \right) \sqrt{\pi} \left( 1 + \operatorname{erf} w \right) \right\}$$

  - Release rate $\dot{N}$
  - Speed ratio $s \equiv \beta u_{\mathrm{e}} = \dfrac{u_{\mathrm{e}}}{\sqrt{2RT_{\mathrm{e}}}}$; $w \equiv s \cos\theta$
  - $A_1$: normalization factor, function of $s$

# Column Number Density (CND, $\sigma$)

- Integrated effect of molecules encountered across a prescribed path $l$
  - When unbounded,

$$\sigma = \int_0^\infty n \, dl$$

- For ISS application, the requirement not to exceed $\sigma_{crit}$ allows one to determine the physical envelope around the source where the limit is violated
  - With a singularity at the source origin, the model will always predict some critical envelope
    - Not consequential for low $\dot{N}$

# Effusive CND Expressions

- For low rate, high-*Kn* venting through an orifice with thermal effusion, no bulk motion, plume model density simplifies to

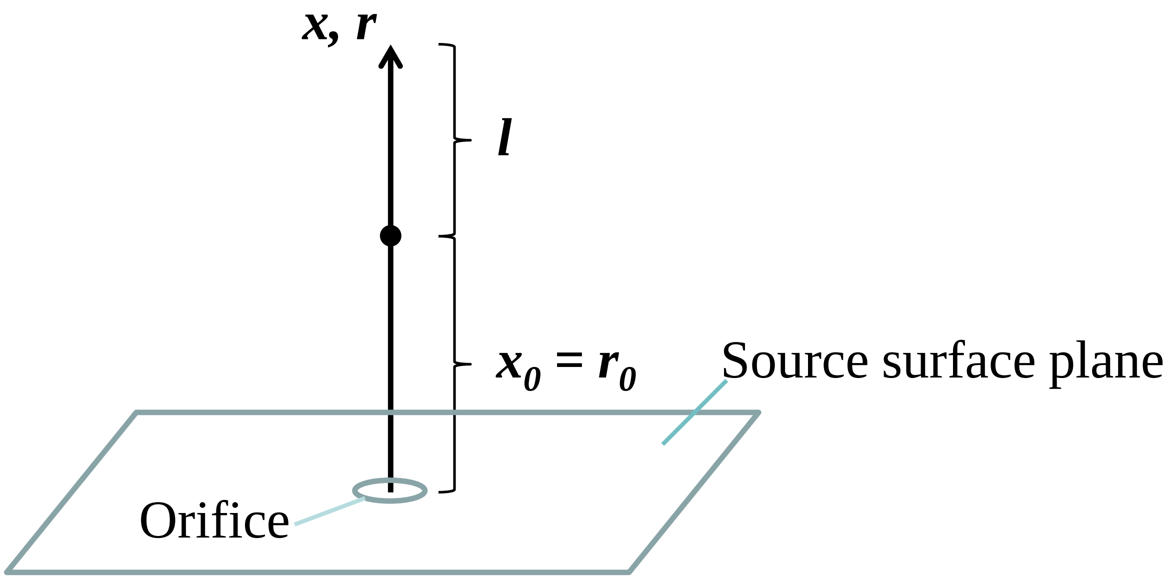
$$n(r,\theta) = \frac{\dot{N}\cos\theta}{r^2\sqrt{8\pi RT}}$$

  - Also describes density field due to outgassing or low rate volatile evaporation from a planar surface viewed from a distance

- Column number density given by

$$\sigma = \frac{\dot{N}}{\sqrt{8\pi RT}}\int_0^\infty \frac{\cos\theta}{r^2}\,dl$$

# 1-D Centerline Path

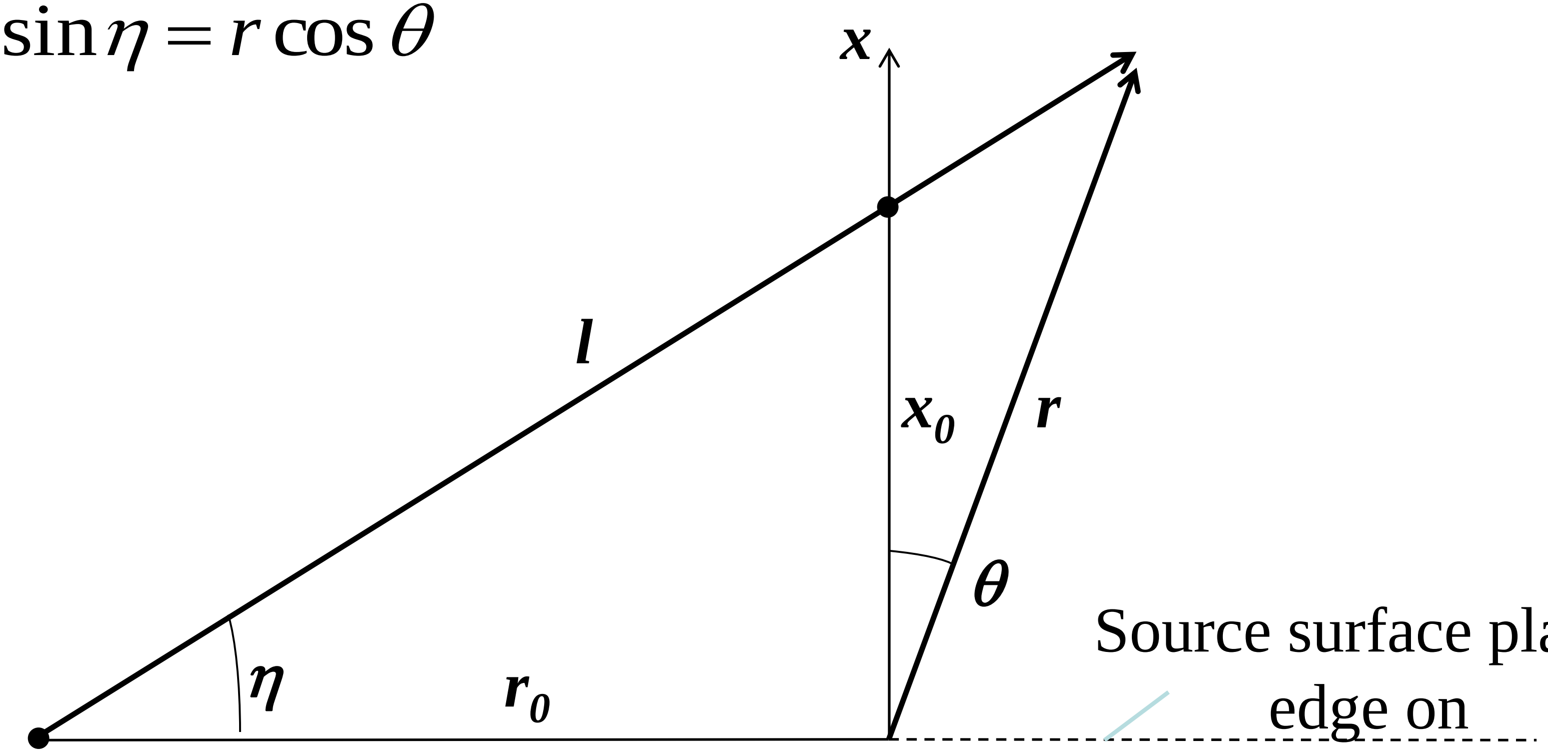



$$l = r - x_0$$

- For effusion, the centerline result is simply $\sigma_{cl,e}(x_0) = \dfrac{\dot{N}}{x_0\sqrt{8\pi RT}}$
- Since density is maximized along centerline, tempting to consider this path produces the highest CND. However, this is not so!

# 2-D Path, Surface Plane Intersection

$$l \sin\eta = r\cos\theta$$



# 2-D Path, Effusion

- Solution for effusion becomes

$$\sigma = \frac{\dot{N}}{\sqrt{8\pi RT}} \frac{\sin\eta}{r_0\left(1-\cos\eta\right)} = \sigma_{cl,e} \frac{\tan\eta\sin\eta}{1-\cos\eta}$$

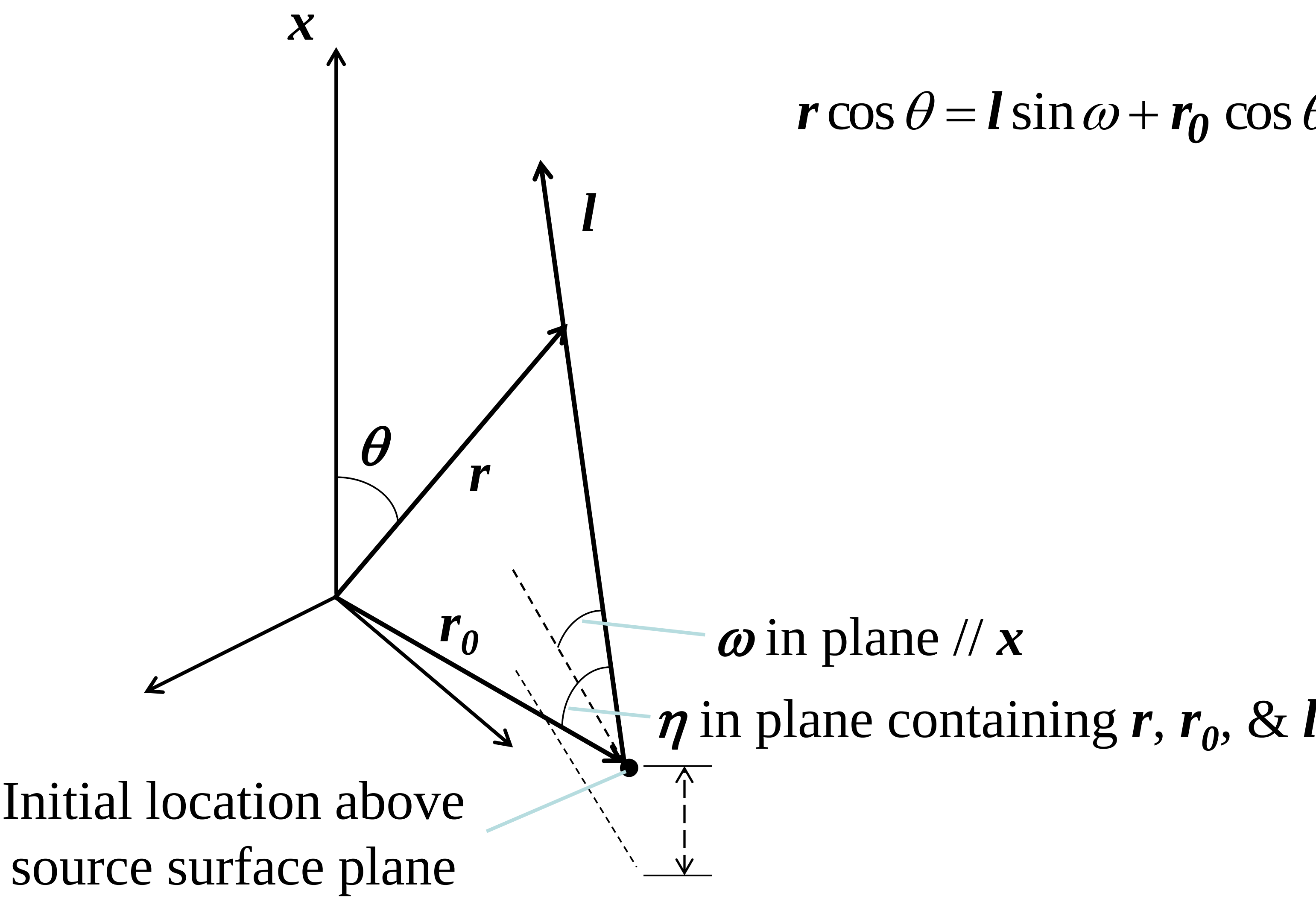
- In the limit where $r_0 \to \infty, \eta \to 0$
  - Vanishingly small distortion of triangle to describe a path parallel to source plane at height $x_0$, find

$$\sigma(\eta \to 0) \to 2\,\sigma_{cl,e}$$

  - Special case may be confirmed by evaluating $\sigma$ along horizontal path at height $x_0$ directly
  - This case provides the maximum CND for effusion

# 3-D General Path

$$\boldsymbol{r}\cos\theta = \boldsymbol{l}\sin\omega + \boldsymbol{r_0}\cos\theta_0$$



# 3-D Path, Effusion

- Effusive gas solution:

$$\sigma = \frac{\dot{N}}{\sqrt{8\pi RT}} \frac{\sin\omega - \cos\theta_0}{r_0\left(1-\cos\eta\right)}$$

- Solution still maximized for distant points along paths parallel to source plane separated by $x_0$
  - Collapses to previous solution

$$\sigma_{\max,e} \rightarrow 2\sigma_{cl,e}$$

# Sonic Orifice Model

- When bulk fluid motion is involved ($s > 0$), plume model behavior becomes too complex to handle directly ($w = s\cos\theta$)

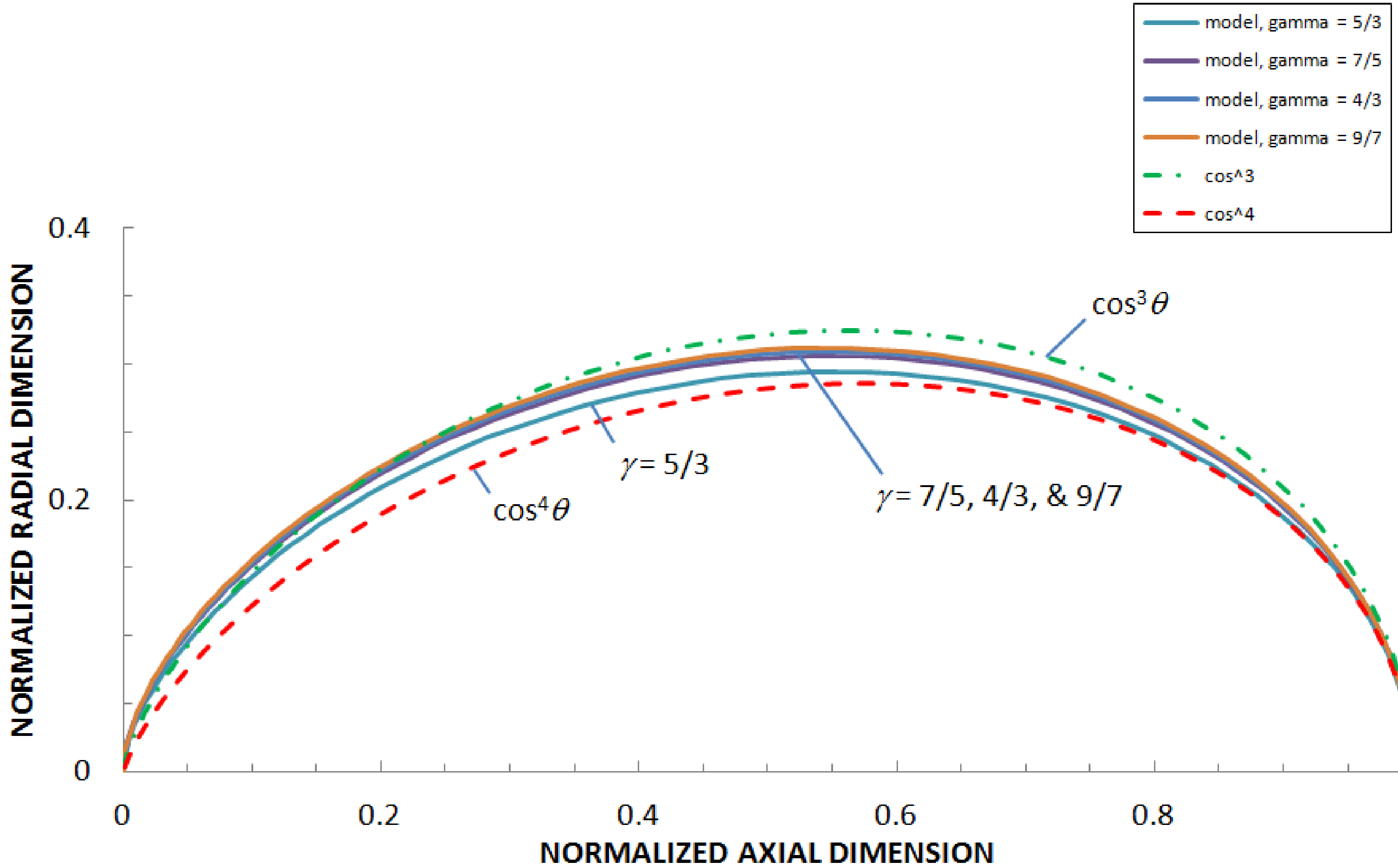
$$n(\boldsymbol{x},t) = \frac{\beta \dot{N} \cos\theta}{A_1 \pi r^2} e^{w^2 - s^2} \left\{ w e^{-w^2} + \left( \frac{1}{2} + w^2 \right) \sqrt{\pi} \left( 1 + \operatorname{erf} w \right) \right\}$$

- Decided to approximate the model behavior, replacing angular distribution by $\cos^3\theta$

$$n_s(r,\theta) \approx K \frac{\cos^3\theta}{r^2}$$

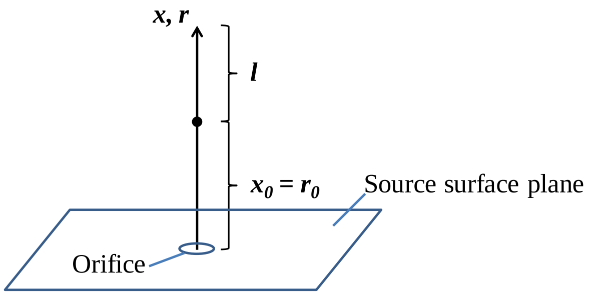
  - Good approximation for many species, different types

# Sonic Angular Distribution Comparison



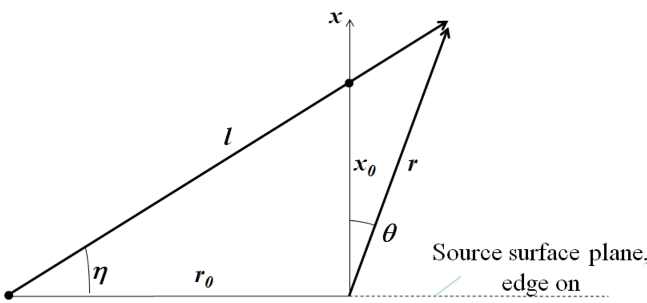
# Some Sonic Model CNDs

- 1-D centerline case: $\sigma_{cl,s} = \dfrac{K}{x_0}$




- 2-D, ∩ centerline & source surface plane:

$$\sigma_s = \frac{K}{3r_0 \sin\eta}\left[2(1+\cos\eta)+\frac{1}{2}\sin\eta\sin 2\eta\right]$$
$$= \frac{\sigma_{cl,s}}{3\cos\eta}\left[2(1+\cos\eta)+\frac{1}{2}\sin\eta\sin 2\eta\right]$$




  – Maximum effect: $\sigma(\eta \to 0) \to \dfrac{4}{3}\sigma_{cl,s}$

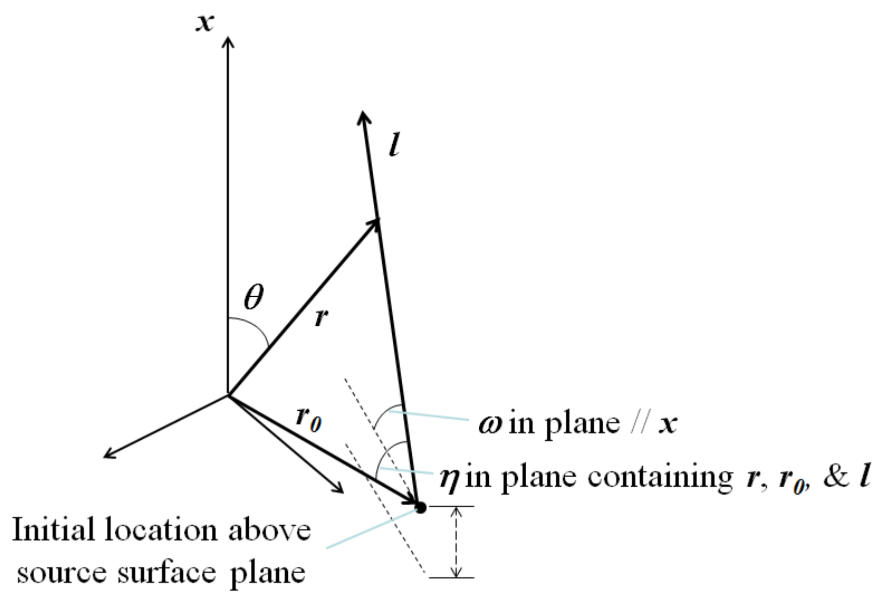
# 3-D Path, Sonic Approximation

- Generally,

$$\sigma = \frac{\sigma_{cl,s}\tan\eta}{3(1-\cos\eta)^2}\left\{\frac{\sin^3\omega(2+3\cos\eta-\cos^3\eta)}{(1+\cos\eta)^2} - 3\sin^2\omega\cos\theta_0 + 3\sin\omega\cos^2\theta_0 - \cos^3\theta_0(2-\cos\eta)\right\}$$

- Maximum effect when

$$\sigma_{\max,s} \to \frac{4}{3}\sigma_{cl,s}$$



# Higher *M* CND Observations

- Assume adequate fit for our purposes using $n(r,\theta) \approx \tilde{K}\dfrac{\cos^m \theta}{r^2}$
- For axial, centerline case, find $\sigma_{cl} = \dfrac{\tilde{K}}{x_0}$
- From previous results, might think limiting transverse case becomes

$$\sigma_{xverse} \stackrel{?}{=} \frac{m+1}{m}\sigma_{cl}$$

  – Always larger than axial

- Actually

$$\sigma_{xverse} = \sigma_{cl} B\left(\frac{1}{2}, \frac{m+1}{2}\right) = \sqrt{\pi}\sigma_{cl}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

  – Axial case is larger for *m* > 5

# Radial Point Source

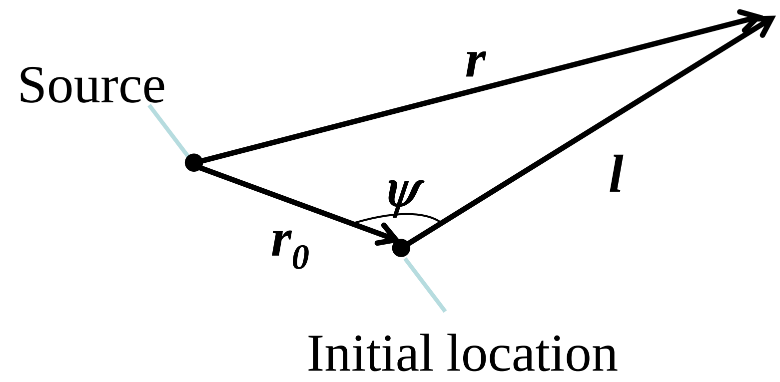
- Model spherically-symmetric expansion
  - No directional constraints
  - No bulk velocity (thermal expansion, *s* = 0)
- Use solution due to Narasimha

$$n_r\left(r\right)=\frac{\dot{N}}{\pi r^2\sqrt{8\pi RT}}$$

# Radial Point Source CND Expressions

- Generally,

$$\sigma = \frac{\dot{N}}{\pi r_0 \sqrt{8\pi RT}} \frac{\pi - \psi}{\sin\psi}$$




  - Notice $r_0 \sin\psi$ acts like $x_0$ in venting cases
- When $\psi = \pi$ (path along source radial line)

$$\sigma_r = \frac{\dot{N}}{\pi r_0 \sqrt{8\pi RT}}$$

- When $\psi = \pi/2$, path begins at right angles to source, $r_0 = x_0$, and

$$\sigma\left(\psi = \frac{\pi}{2}\right) = \frac{\pi}{2}\sigma_r$$

- Maximum CND found for path that extends to infinity in both directions:

$$\sigma_{\max,r} = \pi\,\sigma_r$$ (The $m = 0$ result!)

# Concluding Remarks

- Undertook a study to determine closed form analytical solutions for a number of frequently encountered CND configurations
- For low-rate effusive venting and higher-rate sonic discharges, maximum CNDs should occur along paths parallel to the source plane that intersect the plume axis
- Maximum CNDs for paths immersed in the presence of an unconstrained radial source do not lie along radial trajectories
- For source angular distributions ~$\cos^m \theta$, it was shown for integer values of $m > 5$, maximum CND values switched from transverse to axial paths
  - Likely associated with spacecraft thruster plumes
- These analytical solutions and associated observations should greatly reduce the amount of effort needed to assess CNDs for a variety of space-related applications

# Acknowledgments


- The author gratefully acknowledges support from NASA Contract NNG12CR31C, especially

  - Mr. Benjamin Reed, NASA-GSFC Code 408
  - Ms Kristina Montt de Garcia, NASA-GSFC Code 546
  - Mr. David Hughes, NASA-GSFC Code 546
  - Dr. Dong-Shiun Lin, SGT, Inc.
  - Ms Cori Quirk, SGT, Inc.

# Backup Slides

# Plume Model Formulation—Source

- Find particular solution to collisionless Boltzmann equation for source $Q_1$:

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}}+\boldsymbol{g}\cdot\frac{\partial f}{\partial \boldsymbol{v}}=Q_1$$

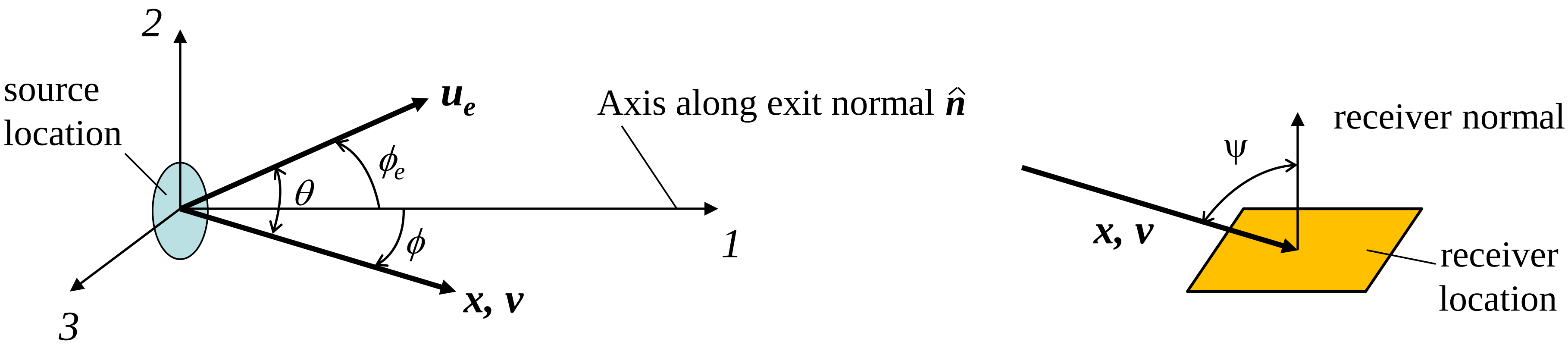
where $Q_1$ represents a Lambertian source superimposed on a bulk velocity

$$Q_1\equiv\frac{2\beta^4}{A_1\pi}\delta(\boldsymbol{x})\dot{m}(t)\left|\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\right|\exp\left(-\beta^2(\boldsymbol{v}-\boldsymbol{u}_\mathrm{e})^2\right)$$

and the normalization factor is given by

$$A_1\equiv e^{-s^2\cos^2\phi_\mathrm{e}}+\sqrt{\pi}\,s\cos\phi_\mathrm{e}\left(1+\mathrm{erf}\left(s\cos\phi_\mathrm{e}\right)\right)$$

# Plume Model Formulation—Definitions




- Subscript *e* represents exit conditions from source
- Simplifies for axisymmetric conditions
  - $\phi_e = 0$
  - $\phi = \theta$
- other definitions: $s \equiv \beta u_e = \dfrac{u_e}{\sqrt{2RT_e}}; \; w \equiv s\cos\theta$